# Non-standard supersymmetries on spaces admitting Killing–Yano tensors



2008 J. Phys. A: Math. Theor. 41 164072

(http://iopscience.iop.org/1751-8121/41/16/164072)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 164072 (8pp) doi:10.1088/1751-8113/41/16/164072

# Non-standard supersymmetries on spaces admitting Killing–Yano tensors

**Mihai Visinescu**

Department of Theoretical Physics, National Institute for Physics and Nuclear Engineering, PO Box M.G.-6, Magurele, Bucharest, Romania

E-mail: mvisin@theory.nipne.ro

Received 18 October 2007, in final form 20 November 2007
Published 9 April 2008
Online at stacks.iop.org/JPhysA/41/164072

**Abstract**

We attempt to point out the intimate relation between Killing–Yano tensors and non-standard supersymmetries, along the way reviewing some recent results. The Euclidean Taub–NUT space is taken to illustrate some of the aspects of the general discussion about the connection between spacetime higher order geometrical symmetries and supersymmetries. The dynamical symmetry of this space leads in a natural way to an infinite dimensional twisted loop superalgebra of the conserved operators.

PACS number: 04.62.+v

## 1. Introduction

In quantum field theories in curved spaces, an important problem is to find the symmetries having geometrical sources able to produce conserved quantities. Spacetime isometries and associated Killing vector fields give rise to constants of motion along geodesics. Besides the ordinary isometries, there are more subtle *hidden* symmetries of the spacetimes encapsulated in higher rank tensors that can occur in association with some supersymmetries.

Let $(M_{n,g})$ be an $n$-dimensional Riemannian manifold, the covariant derivative in the tensor formalism is defined using the Levi-Civita connection and the indices $\mu, \nu, \ldots$ will be raised and lowered with the metric $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$.

**Definition 1.** *A symmetric tensor of* $K_{\mu_1\ldots\mu_r}$ *of rank* $r > 1$ *satisfying a generalized Killing equation*

$$K_{(\mu_1\ldots\mu_r;\lambda)} = 0, \tag{1}$$

*is called a Stäckel–Killing (S–K) tensor.*

Equation (1) ensures that the homogeneous function in momentum $p^\mu$

$$K = K_{\mu_1\ldots\mu_r} p^{\mu_1} \cdots p^{\mu_r}, \qquad p_\mu = g_{\mu\nu}(x)\dot{x}^\nu \tag{2}$$

is a first integral of the geodesic equation, where the over-dot denotes the ordinary proper time derivative.

The next interesting geometrical object connected with higher order symmetries of a manifold after S–K tensors is the Killing–Yano (K–Y) tensors.

**Definition 2.** *A differential p-form $f$ is called a K–Y tensor if its covariant derivative $f_{\mu_1\ldots\mu_p;\lambda}$ is totally antisymmetric.*

Equivalently, a tensor is called a K–Y tensor of valence $p$ if it is totally antisymmetric and satisfies the equation

$$f_{\mu_1\ldots(\mu_p;\lambda)} = 0. \tag{3}$$

Such objects were introduced from a purely mathematical point of view [1], but subsequently it was realized at the intimate connection between K–Y tensors and *non-standard* supersymmetries, both at classical and quantum level.

These two generalizations (1) and (3) of the Killing vector equation could be related. Let $f_{\mu_1\ldots\mu_p}$ be a K–Y tensor, then the tensor field

$$K_{\mu\nu} = f_{\mu\mu_2\ldots\mu_p} f_{\nu}^{\mu_2\ldots\mu_p} \tag{4}$$

is a S–K tensor and it sometimes refers to this S–K tensor as the associated tensor to $f$. However, the converse statement is not true in general: not all S–K tensors of valence 2 are associated to a K–Y tensor.

To illustrate the general results we make use of the Euclidean Taub–NUT space. For a special choice of coordinates the Euclidean Taub–NUT metrics [2] take the form

$$ds_G^2 = f(r)[dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2] + g(r)[d\chi + \cos\theta\,d\varphi]^2 \tag{5}$$

where $r > 0$ is the radial coordinate of $\mathbb{R}^4 - \{0\}$ , the angle variables $(\theta, \varphi, \chi)$, $(0 \leqslant \theta < \pi, 0 \leqslant \varphi < 2\pi, 0 \leqslant \chi < 4\pi)$ parametrize the unit sphere $S^3$. For the generalized Taub–NUT metrics [3] the functions $f(r)$ and $g(r)$ take, respectively, the form

$$f(r) = \frac{a + br}{r}, \qquad g(r) = \frac{ar + br^2}{1 + cr + dr^2}, \tag{6}$$

where $a, b, c$ and $d$ are constants. If the constants $a, b, c$ and $d$ from equation (6) are subject to the constraints $c = \frac{2b}{a}$ and $d = \frac{b^2}{a^2}$ the generalized Taub–NUT metrics [3] coincide, up to a constant factor, with the standard one [2] on setting $4m = \frac{a}{b}$ with $m$ as a real parameter.

Spaces with a metric of the form above have an isometry group $SU(2) \times U(1)$ with four Killing vectors.

As observed in [4], the standard Taub–NUT geometry also possesses four K–Y tensors of valence 2. The first three are rather special, namely they are covariantly constant ($x^i$, $i = 1, 2, 3$ are Cartesian coordinates)

$$\begin{aligned} f^i &= 8m(d\chi + \cos\theta\,d\varphi) \wedge dx^i - \epsilon_{ijk}\left(1 + \frac{4m}{r}\right) dx^j \wedge dx^k, \\ f^i_{\nu\lambda;\mu} &= 0, \qquad i, j, k = 1, 2, 3. \end{aligned} \tag{7}$$

They are mutually anti-commuting and square the minus unity. Thus they are complex structures realizing the quaternion algebra and the standard Taub–NUT manifold is hyper-Kähler.

In addition to the above vector-like K–Y tensors there is also a scalar one

$$f^Y = 8m(d\chi + \cos\theta\,d\varphi) \wedge dr + 4r(r + 2m)\left(1 + \frac{r}{4m}\right)\sin\theta\,d\theta \wedge d\varphi \tag{8}$$

which has a non-vanishing component of the field strength $f^Y_{r\theta;\varphi} \neq 0$.

In the standard Taub–NUT case there is a conserved vector analogous to the Runge–Lenz vector of the Kepler-type problem: $\vec{K} = \frac{1}{2}\vec{K}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. The components $K^i_{\mu\nu}$ involved with the Runge–Lenz-type vector are S–K tensors and they can be expressed as symmetrized products of the K–Y tensors $f^i$ (7) and $f^Y$ (8) [5].

Iwai and Katayama [3] showed that the metric (5) still admits a Kepler-type symmetry if the functions $f(r)$ and $g(r)$ have the form (6). It is a remarkable fact that the S–K tensors involved in the conserved Runge–Lenz vector of the generalized Taub–NUT metrics cannot be expressed in terms of K–Y tensors. These generalizations do not admit K–Y tensors, the only exception being the standard Taub–NUT metric [6].

The paper is organized as follows: first we explain the role of K–Y tensors in connection with non-standard supersymmetries, and the construction of the Dirac-type operators. In section 3 we investigate the new type of symmetries generated by the covariantly constant K–Y tensors that realize certain square roots of the metric tensor. In section 4 we discuss the covariantly non-constant K–Y tensors, their role in the absence of gravitational anomalies and the relations with hidden symmetries. Finally we discuss shortly some problems that deserve further attention.

## 2. Non-standard supersymmetries and Dirac-type operators

The pseudo-classical approach of a spin-$\frac{1}{2}$ fermion in a curved spacetime can be described by the supersymmetric extension of the ordinary relativistic particle [7–9]. The equations of motion of the pseudo-classical Dirac particle can be obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{\mathrm{i}}{2}\eta_{ab}\psi^a\frac{D\psi^b}{D\tau} \tag{9}$$

where the real Grassmann variables $\psi^a$ are introduced to take into account the spin degrees of freedom. Here $\tau$ is the proper time and $\eta_{ab}$ is the flat-space metric. The theory described by the Lagrangian (9) admits *generic* symmetries which exist for any spacetime metric $g_{\mu\nu}$.

For spacetimes admitting K–Y tensors, *non-generic* supersymmetries can be constructed. It has been shown that *non-generic* supercharges can be constructed from K–Y tensors using the formula [8, 10]

$$Q_f = \dot{x}^{\mu_1} f_{\mu_1\ldots\mu_p}\psi^{\mu_2}\cdots\psi^{\mu_p} - \frac{\mathrm{i}}{p+1} f_{\mu_2\ldots\mu_p;\mu_1}\psi^{\mu_1}\cdots\psi^{\mu_{p+1}}. \tag{10}$$

Passing from pseudo-classical description of the fermions to the Dirac theory, we note that for any isometry with a Killing vector $R^\mu$, there is an appropriate operator [11]

$$X_k = -\mathrm{i}\left(R^\mu\nabla_\mu - \tfrac{1}{4}\gamma^\mu\gamma^\nu R_{\mu;\nu}\right), \tag{11}$$

which commutes with the *standard* Dirac operator $D_s$

$$D_s = \mathrm{i}\gamma^\mu\nabla_\mu, \tag{12}$$

where $\nabla_\mu$ are the spin covariant derivatives including spin-connection, while $\gamma^\mu$ are the standard Dirac matrices carrying natural indices.

Moreover each K–Y tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form [11]

$$D_f = \mathrm{i}\gamma^\mu\left(f^\nu_\mu\nabla_\nu - \tfrac{1}{6}\gamma^\nu\gamma^\rho f_{\mu\nu;\rho}\right) \tag{13}$$

which anticommutes with the standard Dirac operator $D_s$ and can be involved in new types of genuine or hidden (super)symmetries.

## 3. Covariantly constant Killing–Yano tensors

Remarkable superalgebras of Dirac-type operators can be produced by special second-order K–Y tensors that represent square roots of the metric tensor.

**Definition 3.** *The non-singular real or complex-valued K–Y tensor $f$ of rank 2 defined on $M_n$ which satisfies*

$$f_\alpha^\mu f_{\mu\beta} = g_{\alpha\beta}, \tag{14}$$

*is called a unit root of the metric tensor of $M_n$, or simply a unit root of $M_n$.*

Let us observe that (14) is a particular case of equation (4) with the metric tensor as an ordinary S–K tensor. It was shown that any K–Y tensor that satisfies equation (14) is covariantly constant [12], i.e. $f_{\mu\nu;\sigma} = 0$.

In what follows we look for manifolds allowing families of unit roots $\mathbf{f} = \{f^i|i = 1, 2, \ldots N_f\}$ having supplementary properties which should guarantee that (I) the linear space $L_\mathbf{f} = \{\rho|\rho = \rho_i f^i, \rho_i \in \mathbb{R}\}$ is isomorphic with a real Lie algebra and (II) each element of $L_\mathbf{f} - 0$ is a root (of arbitrary norm). In these circumstances we have the following theorem [13]:

**Theorem 1.** *The unique type of family of unit roots with $N_f > 1$ having the properties (I) and (II) are the triplets $\mathbf{f} = \{f^1, f^2, f^3\}$ which satisfy*

$$\langle f^i\rangle\langle f^j\rangle = -\delta_{ij} 1_n + \varepsilon_{ijk}\langle f^k\rangle, \qquad i, j, k \ldots = 1, 2, 3. \tag{15}$$

**Proof.** We have denoted by $\langle f\rangle$ the matrix form of the K–Y tensor $f$. Taking into account that $\varepsilon_{ijk}$ is the antisymmetric tensor with $\varepsilon_{123} = 1$ we recognize that equations (15) are the well-known multiplication rules of the quaternion units or similar algebraic structures (e.g. the Pauli matrices). Consequently, the matrices $\langle f^i\rangle$ and $1_n$ generate a matrix representation of the quaternionic algebra $\mathbb{H}$. Other choices are forbidden by the Frobenius theorem. □

If the unit roots $f^i$ have only real-valued components we recover the hypercomplex structures that obey equation (15) and these are connected with the hyper-Kähler structure of the space. An example of hyper-Kähler manifold is the Euclidean Taub–NUT space which is equipped with only one family of real unit roots.

The main geometric feature of all manifolds admitting a triplet of unit roots is given by

**Theorem 2.** *If a manifold $M_n$ allows a triplet of unit roots then this must be Ricci flat (i.e. $R_{\mu\nu} = 0$).*

**Proof.** As in the case of the hyper-Kähler manifolds, we start with the identity $0 = f_{\mu\nu;\alpha;\beta} - f_{\mu\nu;\beta;\alpha} = f_{\mu\sigma} R^\sigma_{\nu\alpha\beta} + f_{\sigma\nu} R^\sigma_{\mu\alpha\beta}$, giving $R_{\mu\nu\alpha\beta} f^\mu_\sigma f^\nu_\tau = R_{\sigma\tau\alpha\beta}$, and calculate

$$R_{\mu\nu\alpha\beta} f^{1\alpha\beta} = R_{\mu\nu\sigma\beta} f^{3\sigma}_\alpha (\langle f^3\rangle\langle f^1\rangle)^{\alpha\beta} = R_{\mu\nu\sigma\beta} f^{3\sigma}_\alpha f^{2\alpha\beta} = -R_{\mu\nu\alpha\beta} f^{1\alpha\beta} = 0.$$

Then, permutating the first three indices of $R$ we find the identity $2R_{\mu\alpha\nu\beta} f^{1\alpha\beta} = R_{\mu\nu\alpha\beta} f^{1\alpha\beta} = 0$. Finally, writing $R_{\mu\nu} = R_{\mu\alpha\nu\beta} f^{1\alpha}_\tau f^{1\beta\tau} = -R_{\mu\alpha\sigma\beta} f^{1\sigma}_\nu f^{1\alpha\beta} = 0$, we draw the conclusion that the manifold is Ricci flat. The same procedure holds for $f^{(2)}$ or $f^{(3)}$ leading to similar identities. □

Note that the manifolds possessing only single unit roots (as the Kähler ones) are not forced to be Ricci flat.

The transition from the complex structures to unit roots has to be productive for the Dirac theory where the complex-valued K–Y tensors could be involved in the theory of the

Dirac-type operators. The covariantly constant K–Y tensor gives rise to Dirac-type operators of the form (13) connected with the standard Dirac operators as follows:

**Theorem 3.** *The Dirac-type operator $D_f$ produced by the K–Y tensor $f$ satisfies the condition*

$$(D_f)^2 = D_s^2 \tag{16}$$

*if and only if $f$ is an unit root.*

**Proof.** The arguments of [12] show that the condition (16) is equivalent to equation (14) with $f$ being a covariantly constant K–Y tensor. □

In particular, referring to the Euclidean Taub–NUT space, from the covariantly constant K–Y tensors $f^i$ (7) we can construct three Dirac-type operators $D^{(i)}$ which anticommute with standard Dirac operator $D_s$ (12). It is convenient to define

$$Q_i = \mathrm{i}H^{-1}D^{(i)}, \tag{17}$$

where $H = -\gamma^0 D_s$ is the massless Hamiltonian operator. These operators form a representation of the quaternionic units

$$Q_i Q_j = \delta_{ij} I + \mathrm{i}\varepsilon_{ijk} Q_k. \tag{18}$$

## 4. Killing–Yano tensors and hidden symmetries

Let us consider a non-trivial S–K tensor of valence 2 with a quadratic constant along the geodesic flow constructed as in equation (2). The generalized Killing equation (1) represents the necessary and sufficient condition for the existence of a quadratic constant of motion as follows from the Poisson bracket of $K$ with the Hamiltonian. Passing from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analog of the quadratic function (2) is [11, 14, 15]

$$\mathcal{K} = \Delta_\mu K^{\mu\nu} \Delta_\nu \tag{19}$$

where $\Delta_\mu$ is the covariant differential operator on the manifold $M_n$. Working out the commutator of (19) with the scalar Laplacian $\mathcal{H} = \Delta_\mu \Delta^\mu$, we get after some calculations

$$[\mathcal{H}, \mathcal{K}] = -\tfrac{4}{3}\left\{K_\lambda^{[\mu} R^{\nu]\lambda}\right\}_{;\nu} \Delta_\mu \tag{20}$$

which means that in general the quantum operator $\mathcal{K}$ does not define a genuine quantum mechanical symmetry [15]. On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S–K tensor $K_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

It is obvious that for a Ricci-flat manifold this quantum anomaly is absent. However, a more interesting situation is represented by the manifolds in which the S–K tensor $K_{\mu\nu}$ can be written as a product of K–Y tensors [11] as in equation (4). On the other hand, the integrability condition for any solution of (3) implies the vanishing of the commutator (20) for S–K tensors which admit a decomposition in terms of K–Y tensors. That is the case of the standard Euclidean Taub–NUT space, but not for his generalizations [3]. In the case of generalized Taub–NUT metrics, there are S–K tensors, but no K–Y tensors and consequently for these spaces there are quantum gravitational anomalies [16].

Usually in $N = 1$ supersymmetric quantum-mechanical models there is a simple H ermitian supercharge $Q$ that close on the Hamiltonian $H$, i.e. $Q^2 = H$. Sometimes it is possible to extend the $N = 1$ supersymmetry to a higher one and the new supercharges close on $H$

$$\{Q^\alpha, Q^\beta\} = 2\delta^{\alpha\beta} H. \tag{21}$$

In the case of the Dirac theory, the new Dirac-type operators constructed from covariantly constant K–Y tensors close on the Klein–Gordon operator, or generally on the square of the standard Dirac operator as in (16).

In the case of hidden symmetries the supercharges anticommute with the original $Q$ but do not necessarily close on $H$. The covariantly non-constant K–Y tensors do not represent 'square roots' of the metric tensor, but generate non-trivial S–K tensor (4).

To make things more specific let us consider the Taub–NUT space which is hyper-Kähler and possesses many non-standard symmetries expressed in terms of four K–Y tensors and three S–K tensors.

K–Y tensors of the Taub–NUT space generate non-generic supersymmetries with supercharges of the form (10) [5, 17–19]. In particular, the supercharge $Q^Y$ corresponding to K–Y tensor $f^Y$ enter the Runge–Lenz vector of the Taub–NUT problem [5].

On the other hand, Dirac-type operator constructed from the K–Y tensor $f^Y$ (8) is $D^Y$ and again it is convenient to define a new operator $Q^Y = HD^Y$.

The conserved Runge–Lenz operator of the Dirac theory is

$$\mathcal{K}_i = \frac{\mu}{4}\{Q^Y, Q_i\} + \frac{1}{2}(\mathcal{B} - P_4)Q_i - \mathcal{J}_i P_4, \tag{22}$$

where $\mathcal{B}^2 = {P_4}^2 - H^2$, $J_i$, $(i = 1, 2, 3)$ are the components of the total angular momentum, while $P_4 = -\mathrm{i}\partial_4$ corresponds to the fourth Cartesian coordinate $x^4 = -4m(\chi + \varphi)$.

The operators $J_i$ and $\mathcal{K}_i$ are involved in the following system of commutation relations:

$$[\mathcal{J}_i, \mathcal{J}_j] = \mathrm{i}\varepsilon_{ijk}\mathcal{J}_k, \qquad [\mathcal{J}_i, \mathcal{K}_j] = \mathrm{i}\varepsilon_{ijk}\mathcal{K}_k, \qquad [\mathcal{K}_i, \mathcal{K}_j] = \mathrm{i}\varepsilon_{ijk}\mathcal{J}_k\mathcal{B}^2, \tag{23}$$

and commute with the operators $Q_i$ (17)

$$[\mathcal{J}_i, Q_j] = \mathrm{i}\varepsilon_{ijk}Q_k, \qquad [\mathcal{K}_i, Q_j] = \mathrm{i}\varepsilon_{ijk}Q_k\mathcal{B}. \tag{24}$$

The algebra (23) does not close as a finite Lie algebra because of the factor $\mathcal{B}^2$. In the standard treatment one concentrates on individual subspaces of the whole Hilbert space which belong to definite eigenvalues of $\mathcal{B}^2$. This is similar to the dynamical algebra of the hydrogen atom [20] which can be identified in a natural way with an infinite dimensional twisted loop algebra.

The dynamical algebras of the Dirac theory have to be obtained by replacing this operator $\mathcal{B}^2$ with its eigenvalue $q^2 - E^2$ and rescaling the operators $\mathcal{K}_i$. The same kind of problems appear for the anticommutators involving the fermionic operators $Q_i$ and $Q^Y$. In what follows, in order to keep the presentation as simple as possible, we shall only give the briefest account of the algebra of operators connected with hidden symmetries in the bosonic sector. For the algebra of operators from the fermionic sector the reader should consult [21].

In the bosonic sector of conserved operators, let us define the new operators 'absorbing' the operator $\mathcal{B}$:

$$J_n^i = \mathcal{J}_i\mathcal{B}^n, \qquad K_n^i = \mathcal{K}_i\mathcal{B}^n, \tag{25}$$

for any $n = 0, 1, 2\ldots$.

Non-trivial commutators of the bosonic sector (23) become

$$\left[J_n^i, J_m^j\right] = \mathrm{i}\varepsilon_{ijk}J_{n+m}^k, \qquad \left[J_n^i, K_m^j\right] = \mathrm{i}\varepsilon_{ijk}K_{n+m}^k, \qquad \left[K_n^i, K_m^j\right] = \mathrm{i}\varepsilon_{ijk}J_{n+m+2}^k. \tag{26}$$

We should like to show that this algebra can be seen as an infinite dimensional twisted loop algebra. The simplest way to achieve a Lie algebra of the Kac–Moody-type is to assign grades to each operators

$$A_{2n}^i := \mathcal{J}_i\mathcal{B}^n, \qquad B_{2n+2}^i := \mathcal{K}_i\mathcal{B}^n. \tag{27}$$

In this way the commutation relations of the bosonic sector are

$$\begin{aligned}
\left[A^i_{2n}, A^j_{2m}\right] &= \mathrm{i}\varepsilon_{ijk} A^k_{2(n+m)},\\
\left[A^i_{2n}, B^j_{2m+2}\right] &= \mathrm{i}\varepsilon_{ijk} B^k_{2(n+m+1)},\\
\left[B^i_{2n+2}, B^j_{2m+2}\right] &= \mathrm{i}\varepsilon_{ijk} A^k_{2(n+m+2)},
\end{aligned} \tag{28}$$

and in this Kac–Moody-type algebra the sum of the grades is conserved under commutation as it is expected.

The same kind of construction can be done in the fermionic sector for the anticommutation relations of the operators $Q_i$, $Q^Y$.

## 5. Concluding comments

To conclude, the Dirac theory in the Euclidean Taub–NUT space gives rise to a large collection of conserved operators associated with standard (Killing vectors) or hidden symmetries (S–K, K–Y tensors). They are involved in interesting and non-trivial algebraic structures as dynamical algebras. The natural way to organize the large collection of conserved operators is to arrange them in a graded loop superalgebra of the Kac–Moody-type. Further work must be done to describe the involution automorphism which is needed to define the twisting in connection with the graded loop algebra of the Kac–Moody-type (28).

We believe that K–Y tensors deserve further attention. They are involved in a multitude of different topics such as conformal S–K or K–Y tensors, non-standard supersymmetries, quantum anomalies, index theorems, etc. So far gravitational anomalies have proved to be absent for scalar fields for spaces admitting K–Y tensors and it would be valuable to know this persist in the case of the full quantum field theories on curved spaces. Concerning the axial anomaly and its connection with the index of the Dirac operators [22–24] the role of K–Y tensors is not obvious. The topological properties of the spaces are more important in comparison with non-standard symmetries.

## Acknowledgments

The author would like to thank Michael Bordag, the organizing committee of the *QFEXT07* workshop for the hospitality and support. He would also like to acknowledge the MEC-CEEX program, Romania, for partial financial support.

## References

[1] Yano K 1952 *Ann. Math.* **55** 328
[2] Hawking S W 1977 *Phys. Lett.* A **60** 81
[3] Iwai T and Katayama N 1993 *J. Geom. Phys.* **12** 55
[4] Gibbons G W and Ruback P J 1987 *Phys. Lett.* B **188** 226
[5] Vaman D and Visinescu M 1998 *Phys. Rev.* D **57** 3790
[6] Visinescu M 2000 *J. Phys. A: Math. Gen.* **33** 4383
[7] Berezin F A and Marinov M S 1975 *JETP Lett.* **21** 320
[8] Gibbons G W, Rietdijk R H and van Holten J-W 1993 *Nucl. Phys.* B **404** 42
[9] Mohseni M 2006 *Int. J. Mod. Phys.* D **15** 121
[10] Tanimoto M 1995 *Nucl. Phys.* B **442** 549
[11] Carter B and McLenaghan R G 1979 *Phys. Rev.* D **19** 1093
[12] Klishevich V V 2000 *Class. Quantum Grav.* **17** 305
[13] Cotăescu I I and Visinescu M 2006 *Fortschr. Phys.* **54** 1142
[14] Carter B 1977 *Phys. Rev.* D **16** 3395

[15] Cariglia M 2004 *Class. Quantum Grav.* **21** 1051
[16] Cotăescu I I, Moroianu S and Visinescu M 2005 *J. Phys. A: Math. Gen.* **38** 7005
[17] van Holten J-W 1995 *Phys. Lett.* B **342** 47
[18] Baleanu D 1994 *Helv. Phys. Acta* **67** 405
[19] Baleanu D 1998 *Gen. Relativ. Gravit.* **30** 195
[20] Daboul J, Slodowy P and Daboul C 1993 *Phys. Lett.* B **317** 321
[21] Cotăescu I I and Visinescu M 2007 *J. Phys. A: Math. Theor.* **40** 11987
[22] van Holten J-W 2003 *Acta Phys. Pol.* B **34** 5983
[23] van Holten J-W 2000 *Annalen Phys.* **9SI** 83 (*Preprint* gr-qc/9910035)
[24] van Holten J-W, Waldron A and Peeters K 1999 *Class. Quantum Grav.* **16** 2537